19-12-19

Δυναμοσειρές γύρω από το x_0: $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ $\quad |x-x_0|<R$

$f(x_0)$ $\quad c_n = \dfrac{f^{n}(x_0)}{n!}$

- $\lim_{n\to\infty} \dfrac{r^n}{n!} \sup |f^{(n)}(x)| = 0$

- $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

- $\dfrac{1}{1-x^2} = \sum_{n=0}^{\infty} x^n \;, |x|<1$

$$\frac{1}{2x+3} = \frac{1}{3(\frac{2}{3}x+1)} = \frac{1}{3}\cdot\frac{1}{1-(-\frac{2}{3}x)} = \frac{1}{3}\sum_{k=0}^{\infty}\left(-\frac{2}{3}x\right)^n$$

$$\sum_{i=k}^{\lambda} f(i) = f(k)+\dots+f(\lambda)$$

$$\kappa\cdot\sum_{n=a}^{\infty} c_n(x-x_0)^n + \lambda\sum_{n=a}^{\infty} b_n(x-x_0)^n = \sum_{\alpha}^{\infty}(\kappa c_n+\lambda b_n)(x-x_0)^n.$$

$$\left[\sum_{n=0}^{\infty} c_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right] = \sum_{n=0}^{\infty} d_n(x-x_0)^n$$

όπου $d_5 = c_0b_5 + c_1b_4 + c_2b_3 + c_3b_2 + c_4b_1 + c_5b_0$

$$d_n = \sum_{i=0}^{n} c_i b_{n-i}$$

$$\left[\sum_{n=0}^{\infty} c_n(x-x_0)^n\right]' = \left[c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots +\right]'$$

$$= 0 + \underbrace{c_1 + c_2\cdot 2(x-x_0) + c_3\cdot 3\cdot(x-x_0)^2 + \dots + c_n\cdot n(x-x_0)^{n-1} + \dots}_{\sum^{\infty} n c_n(x-x_0)^{n-1}}$$

(E) $\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = 0$ $\quad \alpha_0, \alpha_1, \alpha_2 \in C(I)$, $x_0 \in I$.

$\alpha_2 \neq 0$ στο I

x_0: ομαλό σημείο $\to$ $\alpha_2(x_0) \neq 0$

$\to$ $\frac{\alpha_1}{\alpha_2}$, $\frac{\alpha_0}{\alpha_2}$ αναλυτικές στο x_0.

ΘΕΩΡΗΜΑ: Ας είναι $x_0 \in I$ ένα ομαλό σημείο της εξ. (E) και $\sum_{n=0}^{\infty} p_n (x-x_0)^n$, $\sum_{n=0}^{\infty} q_n (x-x_0)^n$ δυο δυναμοσειρές με ακτίνα σύγκλισης και τέτοιες ώστε

$\frac{\alpha_1(x)}{\alpha_2(x)} = \sum_{n=0}^{\infty} p_n (x-x_0)^n$, $|x-x_0| < R_1$, $\quad \frac{\alpha_0(x)}{\alpha_1(x)} = \sum_{n=0}^{\infty} q_n (x-x_0)^n$ $\quad |x-x_0| < R_2$

αν $c_0, c_1 \in \mathbb{R}$ τότε υπάρχουν $c_n \, (n \geq 2)$ έτσι ώστε η δυναμοσειρά $\sum_{n=0}^{\infty} c_n (x-x_0)^n$ να έχει ακτίνα σύγκλισης

τουλάχιστον $R = \min\{R_1, R_2\}$ και η συνάρτηση $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$, $|x-x_0| < R$ να είναι λύση της

(E) που ικανοποιεί τις αρχικές συνθήκες $c_0 = y(x_0)$, $c_1 = y'($

Παράδειγμα: Να επιλυθεί η εξ. γύρω από το σημείο $x_0 = 1$

$y'' - 2(x-1) y' + 2y = 0$, $x_0 = 1$

$\alpha_2(x) = 1$, $\alpha_1(x) = -2(x-1)$, $\alpha_0(x) = 2$

$\alpha_2(1) \neq 0$.

$\frac{\alpha_1(x)}{\alpha_2(x)} = -\frac{2(x-1)}{1} = -2(x-1)$, $R_1 = +\infty$.

$\frac{\alpha_0(x)}{\alpha_2(x)} = \frac{2}{1} = 2$, $R_2 = +\infty$.

$R_1 = +\infty = R_2 \to \boxed{R = +\infty}$ $\quad$ (c_0, c_1 δεδομένα)

$y'(x) = \sum_{n=1}^{\infty} c_n n (x-x_0)^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} c_n n(n-1)(x-x_0)^{n-2}$

$$0=\sum_{n=2} c_n n(n-1)(x-1)^{n-2} - 2(x-1)\sum_{n=1}^{\infty} c_n n(x-1)^{n-1} + 2\cdot\sum_{n=0}^{\infty} c_n (x-1)^n$$

$$0=\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} - \sum_{n=1}^{\infty} 2\cdot n c_n (x-1)^n + \sum_{n=0}^{\infty} 2c_n(x-1)^n.$$

$$0=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}(x-1)^n - \sum_{n=1}^{\infty} 2nc_n(x-1)^n + \sum_{n=0}^{\infty} 2c_n(x-1)^n.$$

$$0=2\cdot1\cdot c_2+\sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}(x-1)^n - \sum_{n=1}^{\infty} 2nc_n(x-1)^n$$

$$+2c_0+\sum_{n=1}^{\infty} 2c_n(x-1)^n$$

$$0=2(c_0+c_2)+\sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2}-2nc_n+2c_n\right](x-1)^n,\ x\in\mathbb{R}.$$

Πρέπει:

$$\begin{cases} c_0+c_2=0 \\ (n+2)(n+1)c_{n+2}-2nc_n+2c_n=0,\ n=1,2,\dots \end{cases}$$

$$\Rightarrow\begin{cases} c_2=-c_0. \\ c_{n+2}=\dfrac{2(n-1)}{(n+1)(n+2)}c_n,\ n=1,2,\dots \end{cases}$$

• $\boxed{\text{Αν } n \text{ άρτιος}}$ $n=2k$ τότε.

$$c_{2k+2}=\frac{2(2k-1)}{(2k+1)(2k+2)}\cdot c_{2k},\ n=1,2,\dots$$

$\boxed{c_2=-c_0}$, $k=1$ $c_4=\dfrac{2\cdot1}{3\cdot4}c_2$ / $k=2$ $c_6=\dfrac{2\cdot3}{5\cdot6}c_4$

$k=3$: $c_8=\dfrac{2\cdot5}{7\cdot8}c_6$, ..., $k=v-1$ $c_{2v}=\dfrac{2(2v-3)}{(2v-1)2v}c_{2v-1}$.

Πολλ/ντας κατά μέλη θα έχουμε.

$$c_{2v}=\frac{-c_0\cdot2^{v-1}(1\cdot3\cdot5\cdots(2v-3))}{[(3\cdot5\cdot7\cdots(2v-1)][4\cdot6\cdot8\cdots2v]},\ v\geq1$$

$$c_{2v}=-c_0\cdot\frac{2^{v-1}}{(2v-1)2^{v-1}(2\cdot3\cdots v)}=-c_0\cdot\frac{1}{(2v-1)(v!)},\ v\geq1$$

• $\boxed{\text{Αν } n \text{ περιττός}}$ $n = 2k+1 \geq 1 \Rightarrow k = 0, 1, 2, \dots$

$$C_{2k+3} = \frac{2\cdot(2k)}{(2k+2)(2k+3)} \cdot C_{2k+1}, \quad k = 0, 1, 2, \dots$$

• $\boxed{k=0}$ $C_3 = 0$

$\boxed{k=1}$ $C_5 = 0$

$k=2$ $C_7 = 0$

$\Big\}$ $C_{2k+1} = 0, \; k \geq 0.$

$$y(x) = \sum_{n=0}^{\infty} C_n (x-1)^n = C_0 + C_1 x + \sum_{n=1}^{\infty} C_{2n}(x-1)^{2n} + \sum_{n=1}^{\infty} \cancel{C_{2n+1}(x-1)^{2n+1}} \to 0$$

$$y(x) = C_0 + C_1(x-1) - C_0 \sum_{n=0}^{\infty} \frac{1}{n!(2n-1)} (x-1)^{2n}, \quad x \in \mathbb{R}$$

$$y(x) = C_0 \cdot \left[1 - \sum_{n=1}^{\infty} \frac{1}{n!(2n-1)} (x-1)^{2n}\right] + C_1(x-1) \quad x \in \mathbb{R}$$

↑ τύπος που δίνει όλες τις λύσεις.

Παίρνουμε 2 λύσεις $\quad y(x_0) = C_0 = \begin{array}{cc} y_1 & y_2 \\ 1 & 0 \end{array}$

$y'(x_0) = C_1 = \begin{array}{cc} 0 & 1 \end{array}$

άρα $y_1(1) = 1$ ή $y_1'(0) = 0$

$y_2(1) = 0$ $\quad y_2'(1) = 1$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0.$$

<u>Παράδειγμα 3</u> $(1-x)y'' - y' + xy = 0$, $y(0) = 1$, $y'(0) = 1$

$x_0 = 0.$ $\qquad\qquad y(0) = C_0, \; y'(0) = C_1$

$\alpha_2(x) = 1 - x$, $\alpha_1(x) = -1$ $\quad \alpha_0(x) = -x$

$\alpha_2(0) = -1 \neq 0$ $\quad , \dfrac{\alpha_1(x)}{\alpha_2(x)} = \dfrac{1}{1-x}$, $|x| < 1 \to R_1 = 1.$

$\dfrac{\alpha_0(x)}{\alpha_2(x)} = -\dfrac{x}{1-x}$, $|x| < 1$ $\quad R_2 = 1$ $\quad , \boxed{R = 1}$

$$0 = (1-x)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} nC_n x^{n-1} + x\sum_{n=0}^{\infty} C_n x^n.$$

$$0 = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)C_n x^{n-1} - \sum_{n=1}^{\infty} + \sum_{n=0}^{\infty} C_n x^{n+1}$$

$$= \sum_{n=1}^{\infty} (n+1)(n)\, c_{n+1} x^{n-1} - \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-1} - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=2}^{\infty} c_{n-2} x^{n-1}$$

$$= 2\cdot 1\cdot c_2 x^0 - c_1 x^0 + \sum_{n=2}^{\infty} (x-1)^{n-1} \left[n(n+1)c_{n+1} - n\cdot(n-1)c_n - n c_n + c_{n-2} \right] = 0.$$

$\boxed{2c_2 = c_1}$ $\quad n(n+1)c_{n+1} = c_n(n^2 - n + n] - c_{n-2}$

$$\boxed{n\cdot(n+1)\, c_{n+1} = n c_n - c_{n-2}\cdot 1 \quad , n = 2, 3, \dots}$$

$c_0 = 1$, $c_1 = 1$, $c_2 = 1/2$, $c_{n+1} = \frac{1}{(n+1)\cdot n}\left[n^2 c_n - c_{n-2} \right]$, $n = 2, 3, \dots$

$n=2$, $c_3 = \frac{1}{3\cdot 2}\left(4\cdot\frac{1}{2} - 1\right) = \frac{1}{2\cdot 3}$ $\quad c_n = \frac{1}{n!}$

$n=3$, $c_4 = \frac{1}{2\cdot 3\cdot 4}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x.$

$n=4$, $c_5 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$

$y'' - xy = 0$, $x_0 = 0.$

$\alpha_2 = 1$, $\alpha_1 = 0$, $\alpha_0 = -x$ $\quad \frac{\alpha_1}{\alpha_2} = 0$ $\quad \frac{\alpha_0}{\alpha_2} = -x$ $\quad R = +\infty.$

2ου βαθμου →

= 2101

c_0, c_1 δεδομενα, $c_2 = 0.$

$(n+2)(n+1)c_{n+2} - c_{n-1} = 0$, $n = 1, 2, \dots$

- $\nu = 3k \geq 1$, $k = 1, 2, 3.$

$(3k+2)(3k+1)c_{3k+2} = c_{3k-1}$ $\quad k = 1, 2 \dots$ $\quad c_2 = 0 = c_8 = c_{11} = c_5 = 0.$

- $\nu = 3k+1$, $3k+1 \geq 1$, $k = 0, 1, 2, \dots$

$c_{3k+3} = \frac{1}{(3k+1)(3k+2)} \cdot c_{3k}$, $\quad c_3 = \frac{1}{3\cdot 2} c_0.$

- $\nu = 3k+2$ (ομοίως)

Παράδειγμα 5: $x^3(x^2-1)y'' + x(x^2-1)y' + y = 0$ << X

$(, +\infty)$

$t = \frac{1}{x}$ $\quad y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dx}{dt} = -t^2 \frac{dy}{dt} = y'$

$y'' = \frac{d^2}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x^2}\frac{dy}{dt}\right) = (-2)(-1)\frac{1}{x^3}\frac{dy}{dt} + \left(-\frac{1}{x^2}\right)\frac{d}{dt}\frac{dy}{dt}\frac{dt}{dx}$

$y'' = 2t^3\frac{dy}{dt} - t^2\frac{d^2y}{dt^2}$

$(1-t^2)\frac{d^2y}{dt^2} - t\frac{dy}{dt} + y = 0, \quad \boxed{t_0 = 0}$

$|t|<1$

$\rightarrow y_1(t) = t \quad , y_2(t) = 1 - \frac{t^2}{2} - \Sigma(\ )t^{2n}$

$y_1(x) = 1/x$ $\quad \frac{1}{|x|} < 1 \Rightarrow |x| > 1$

$y_2(x) = 1 - \frac{1}{2x^2} - \Sigma(\ )\frac{1}{x^{2n}} \quad , x > 1$